B.Sc-part-II (Sub). Diff. Calculus. 09-07-2020.

Q.1. State and prove Maclaurin's theorem to expand $f(x)$.

Soln: Statement:→ Let the function $f(x)$ can be expanded in powers of x, then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \cdots \text{ to } \infty.$$

Proof: Let $f(x)$ in a convergent series of positive integral powers of x, so that

$$f(x) = A_0 + A_1 x + A_2 \frac{x^2}{\lfloor 2} + A_3 \frac{x^3}{\lfloor 3} + A_4 \frac{x^4}{\lfloor 4} + \cdots \text{ to } \infty \quad — (1)$$

Where, $A_0, A_1, A_2, \ldots$ are constants to be determined, not containing x

Differentiating (1) with respect to x, we get

$$f'(x) = A_1 + A_2 \frac{2x}{\lfloor 2} + A_3 \frac{3x^2}{\lfloor 3} + A_4 \frac{4x^3}{\lfloor 4} + \cdots \text{ to } \infty.$$

$$\Rightarrow f'(x) = A_1 + A_2 x + A_3 \frac{x^2}{\lfloor 2} + A_3 \frac{x^2}{\lfloor 2} + A_4 \frac{x^3}{\lfloor 3} + \cdots \text{ to } \infty \quad — (2)$$

Differentiating (2) with respect to x, we get

$$f''(x) = A_2 + A_3 \frac{2x}{\lfloor 2} + A_4 \frac{3x^2}{\lfloor 3} + \cdots \text{ to } \infty.$$

$$\Rightarrow f''(x) = A_2 + A_3 x + A_4 \frac{x^2}{\lfloor 2} + \cdots \text{ to } \infty \quad (3)$$

Diff. (3) with respect to x, we get

$$f'''(x) = A_3 + A_4 \frac{2x}{\lfloor 2} + \cdots \text{ to } \infty$$

$$\Rightarrow f'''(x) = A_3 + A_4 x + \cdots \text{ to } \infty \quad — (4)$$

Putting $x=0$ in (1), (2), (3), (4), ... we get

$f(0) = A_0$, $f'(0) = A_1$, $f''(0) = A_2$; $f'''(0) = A_3$, ...

Substituting the values of these constants in (1), we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \cdots \text{ to } \infty.$$

Proved

Unit. III Diff. Calculus 09-07-2020.

Problems on Maclaurin's Series.

Q. 2. Apply Maclaurin's series to prove that

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \dots \text{ to } \infty.$$

Soln: Let $f(x) = \sin x$

Diff. successively w.r.t. to x, $f'(x) = \cos x$

$f''(x) = -\sin x$; $f'''(x) = -\cos x$, $f^{iv}(x) = \sin x$

$f^{v}(x) = \cos x$ and so on.

At $x = 0$, $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1$

$f^{iv}(0) = 0$ $f^{v}(0) = 1$, and so on.

By Maclaurin's series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \frac{x^4}{\lfloor 4} f^{iv}(0)$$

$$+ \frac{x^5}{\lfloor 5} f^{v}(0) + \dots \text{ to } \infty.$$

Hence, $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \dots \text{ to } \infty.$

Q. 3. Expand $e^{\sin x}$ as far as the term involving x^4.

Soln Let $f(x) = e^{\sin x}$ $\therefore f(0) = e^{\sin 0} = e^0 = 1$

and $f'(x) = e^{\sin x} \cos x$ $\therefore f'(0) = 1$

$f''(x) = e^{\sin x} \cos^2 x - e^{\sin x} \sin x$ $\therefore f''(0) = 1$

$f'''(x) = e^{\sin x} \cos^3 x - e^{\sin x}\, 2 \cos x \sin x - e^{\sin x} \sin x \cos x - e^{\sin x} \cos x$

$= e^{\sin x} \cos^3 x - \frac{3}{2} e^{\sin x} \sin 2x - e^{\sin x} \cos x$

$\therefore f'''(0) = 0$

$f^{iv}(0) = e^{\sin x} \cos^4 x - e^{\sin x} \cdot 3 \cos^2 x \cdot \sin x$

$- \frac{1}{2} e^{\sin x} \cos x \sin 2x - \frac{3}{2} e^{\sin x} \cos 2x \cdot 2 - e^{\sin x} \cos^2 x + e^{\sin x} \sin x$

$\therefore f^{iv}(0) = -3$

Hence, from the Maclaurin's Theorem

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \frac{x^4}{\lfloor 4} f^{iv}(0)$$

We get $e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \dots \text{ to } \infty.$

Solved.